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Analysing Temporal Reasoning in Description Logics Using Formal Grammars (IJCAI Extended Abstract)*

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Abstract

We establish a correspondence between (fragments of) $\mathcal{TEL}^{\bigcirc}$, a temporal extension of the $\mathcal{EL}$ description logic with the LTL operator $\bigcirc^k$, and some specific kinds of formal grammars, in particular, conjunctive grammars (context-free grammars equipped with the operation of intersection). This connection implies that $\mathcal{TEL}^{\bigcirc}$ does not possess the property of ultimate periodicity of models, and further leads to undecidability of query answering in $\mathcal{TEL}^{\bigcirc}$, closing a question left open since the introduction of $\mathcal{TEL}^{\bigcirc}$. Moreover, it also allows to establish decidability of query answering for some new interesting fragments of $\mathcal{TEL}^{\bigcirc}$, and to reuse for this purpose existing tools and algorithms for conjunctive grammars.

1 Introduction

Ontology-mediated query answering (OMQA) aims at improving data access by enriching data with domain knowledge. This knowledge, typically expressed in a given description logic (DL) as a set of logical formulas (concept inclusions, in DL parlance), enables comprehensive query answering based not only on facts stored in the data but also on facts deducible through logical reasoning [Poggi *et al.*, 2008; Kontchakov and Zakharyaschev, 2014; Bienvenu and Ortiz, 2015; Baader *et al.*, 2003]. In practical applications, where the focus is on efficiency, many ontologies use DLs from the $\mathcal{EL}$ family [Baader *et al.*, 2005; Baader *et al.*, 2008], which allow for tractable reasoning.

Various extensions of the OMQA framework have been proposed to make it applicable to querying temporal data [Artale *et al.*, 2017]. In this setting, facts are associated with *timestamps*, and concept inclusions may feature some operators from *linear temporal logic* (*LTL*). We consider the temporal description logic $\mathcal{TEL}$, which extends $\mathcal{EL}$ with the operators $\bigcirc$ (next), $\bigcirc^-$ (previous), $\Diamond$ (eventually) and $\Diamond^-$ (eventually in the past) [Artale *et al.*, 2007; Gutiérrez-Basulto *et al.*, 2016]. For instance, the concept inclusion $\mathsf{Prof} \sqsubseteq \bigcirc\mathsf{Prof}$ means that, at any moment, someone that is a professor is also a professor at the next instant. Moreover, $\mathcal{TEL}$ allows the user to specify that some relations between individuals (called 'roles') are *rigid*, i.e., they do not change over time.

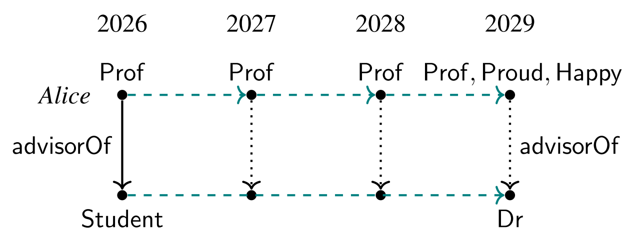

Figure 1: Representation of some inferences for Example 1. Dashed lines represent the temporal evolution of a given element, and dotted lines a relation whose existence is known due to role rigidity.

Example 1. *Imagine that Alice is a professor in 2026, denoted* $\mathsf{Prof}(\mathrm{Alice}, 2026)$. *Professorship is permanent and requires advising students, who in three years become doctors. Being an advisor of a doctor makes one proud, and proud professors are happy. This knowledge is formalised as follows (using a rigid role* $\mathsf{advisorOf}$*):*

$$\mathsf{Prof} \sqsubseteq \bigcirc\mathsf{Prof} \quad \mathsf{Prof} \sqcap \mathsf{Proud} \sqsubseteq \mathsf{Happy} \quad \mathsf{Student} \sqsubseteq \bigcirc^3\mathsf{Dr}$$
$$\mathsf{Prof} \sqsubseteq \exists\mathsf{advisorOf}.\mathsf{Student} \quad \exists\mathsf{advisorOf}.\mathsf{Dr} \sqsubseteq \mathsf{Proud}$$

Figure 1 provides a graphical representation of some information about Alice that can be inferred from $\mathsf{Prof}(\mathrm{Alice}, 2026)$ *and the above* $\mathcal{TEL}$ *TBox. In particular, Alice is happy at year* 2029.

Despite both *LTL* and $\mathcal{EL}$ being decidable, Gutiérrez-Basulto *et al.* [2016] showed that their combination in $\mathcal{TEL}$ quickly leads to undecidability. They exhibited fragments of $\mathcal{TEL}$ that allow for decidable query answering by guaranteeing a certain semantic property: the existence of models that are *ultimately periodic*. In a nutshell, a model is ultimately periodic if the evolution over time of any given element is, after some initial segment, periodic. It was shown that the use of the operator $\Diamond$ destroys this property, while it is enjoyed by certain 'acyclic' fragments of $\mathcal{TEL}^{\bigcirc}$, a sublanguage limited to operators $\bigcirc$ and $\bigcirc^-$. The question of ultimate periodicity, and, in general, of decidability of query answering for the full $\mathcal{TEL}^{\bigcirc}$ was left open.

*For the full version, see Bourgaux *et al.* [2025a; 2025b]

Our main contribution is to link temporal reasoning in $\mathcal{TEL}^{\bigcirc}$ to the study of associated formal languages, in particular, conjunctive languages of Okhotin [2013]. This connection allows us to close the open question of Gutiérrez-Basulto *et al.* [2016] and obtain additional results. First, we close negatively the question of whether $\mathcal{TEL}^{\bigcirc}$ enjoys ultimate periodicity, and complete the complexity picture of $\mathcal{TEL}$: atomic query answering in $\mathcal{TEL}^{\bigcirc}$ is undecidable. Second, we provide results for two new fragments $\mathcal{TEL}^{\bigcirc}_{fut}$ and $\mathcal{TEL}^{\bigcirc}_{lin}$ we introduce. As for $\mathcal{TEL}^{\bigcirc}_{fut}$, which only allows to derive novel information about the future (or present) and not the past, we prove that atomic query answering is solvable in polynomial time (in combined and data complexity). Regarding $\mathcal{TEL}^{\bigcirc}_{lin}$, a temporal extension of linear $\mathcal{EL}$ [Dimartino *et al.*, 2016], we prove that $\mathcal{TEL}^{\bigcirc}_{lin}$ enjoys ultimate periodicity, and derive complexity results for atomic query answering. For details and proofs, we refer the reader to the main paper [Bourgaux *et al.*, 2025a; 2025b]. We also point to a recent continuation of this line of work which generalises our results [Gnatenko and Kontchakov, 2026].

2 Preliminaries

We assume familiarity with formal languages, grammars, and semilinear sets. We recall the definition of $\mathcal{TEL}^{\bigcirc}$ from Gutiérrez-Basulto *et al.* [2016]. Let $\mathsf{N_I}$, $\mathsf{N_C}$, $\mathsf{N_R}$ be disjoint countably infinite sets of *individual names*, *concept names* and *role names*, respectively, with $\mathsf{N_R}$ partitioned into *rigid role names* $\mathsf{N_R^{rig}}$ and *local role names* $\mathsf{N_R^{loc}}$.

Syntax. A *fact* is an expression of the form $A(a, n)$ or $r(a, b, n)$, where $a, b \in \mathsf{N_I}$, $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and $n \in \mathbb{Z}$. An *ABox* (data instance) $\mathcal{A}$ is a finite set of facts. A $\mathcal{TEL}^{\bigcirc}$-*TBox* (ontology) $\mathcal{T}$ is a finite set of *concept inclusions* of the form

$$A \sqsubseteq \bigcirc^n B \quad A \sqcap A' \sqsubseteq B \quad \exists r.A \sqsubseteq B \quad A \sqsubseteq \exists r.B \qquad (1)$$

where $A, A', B \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and $n \in \mathbb{Z}$. When $n = 0$, $n = 1$, or $n = -1$, we simply write $A \sqsubseteq B$, $A \sqsubseteq \bigcirc B$, and $A \sqsubseteq \bigcirc^- B$, respectively. A *knowledge base (KB)* is a pair $(\mathcal{T}, \mathcal{A})$. We denote by $\mathsf{N_C}(\mathcal{T})$, $\mathsf{N_R}(\mathcal{T})$, $\mathsf{N_R^{rig}}(\mathcal{T})$, and $\mathsf{N_R^{loc}}(\mathcal{T})$, respectively, the sets of concept names, role names, and rigid and local role names appearing in $\mathcal{T}$. The *size* $|\mathcal{T}|$ of $\mathcal{T}$ (resp. $|\mathcal{A}|$ of $\mathcal{A}$) is the number of symbols needed to write it down, with integers encoded in *unary*.

We will further consider two fragments of $\mathcal{TEL}^{\bigcirc}$: the *future* fragment, $\mathcal{TEL}^{\bigcirc}_{fut}$, is obtained by setting $n \geqslant 0$, and the *linear* fragment, $\mathcal{TEL}^{\bigcirc}_{lin}$, disallows concept inclusions of the form $A \sqcap A' \sqsubseteq B$. For $\mathcal{TEL}^{\bigcirc}_{lin}$, moreover, we distinguish the case where all role names in TBoxes are rigid; the corresponding subfragment is denoted $\mathcal{TEL}^{\bigcirc}_{lin\&rig}$.

Semantics. An *interpretation* $\mathfrak{J}$ is a structure $(\Delta^{\mathfrak{J}}, (\mathcal{I}_i)_{i\in\mathbb{Z}})$ where each $\mathcal{I}_i = (\Delta^{\mathfrak{J}}, \cdot^{\mathcal{I}_i})$ is a classical DL interpretation with domain $\Delta^{\mathfrak{J}}$: for every $a \in \mathsf{N_I}$, $a^{\mathcal{I}_i} = a$ (*standard name assumption*, assuming that $\mathsf{N_I} \subseteq \Delta^{\mathfrak{J}}$), for every $A \in \mathsf{N_C}$, $A^{\mathcal{I}_i} \subseteq \Delta^{\mathfrak{J}}$, and for every $r \in \mathsf{N_R}$, $r^{\mathcal{I}_i} \subseteq \Delta^{\mathfrak{J}} \times \Delta^{\mathfrak{J}}$. Moreover, for every $r \in \mathsf{N_R^{rig}}$, $r^{\mathcal{I}_i} = r^{\mathcal{I}_0}$ for every $i \in \mathbb{Z}$. The interpretation function $\cdot^{\mathcal{I}_i}$ is often written as $\cdot^{\mathfrak{J},i}$ and is extended to interpret complex concepts as expected:

$$(\bigcirc^n A)^{\mathfrak{J},i} = A^{\mathfrak{J},i+n} \qquad (A \sqcap B)^{\mathfrak{J},i} = A^{\mathfrak{J},i} \cap B^{\mathfrak{J},i}$$
$$(\exists r.A)^{\mathfrak{J},i} = \{d \mid \exists e \in A^{\mathfrak{J},i}, (d, e) \in r^{\mathfrak{J},i}\}$$

The interpretation $\mathfrak{J}$ is a *model* of a fact $A(a, n)$ (resp. $r(a, b, n)$) if $a \in A^{\mathfrak{J},n}$ (resp. $(a, b) \in r^{\mathfrak{J},n}$), and of a concept inclusion $C \sqsubseteq D$ if $C^{\mathfrak{J},i} \subseteq D^{\mathfrak{J},i}$ for every $i \in \mathbb{Z}$. It is a model of an ABox $\mathcal{A}$ (resp. a TBox $\mathcal{T}$) if it is a model of all facts in $\mathcal{A}$ (resp. all concept inclusions in $\mathcal{T}$), and of a KB $(\mathcal{T}, \mathcal{A})$ if it is a model of $\mathcal{T}$ and $\mathcal{A}$. We write $\mathfrak{J} \models \alpha$ to denote that $\mathfrak{J}$ is a model of α. A TBox $\mathcal{T}$ *entails* a concept inclusion α, written $\mathcal{T} \models \alpha$, if $\mathfrak{J} \models \mathcal{T}$ implies $\mathfrak{J} \models \alpha$, and a KB $(\mathcal{T}, \mathcal{A})$ entails a fact α, $(\mathcal{T}, \mathcal{A}) \models \alpha$, if $\mathfrak{J} \models (\mathcal{T}, \mathcal{A})$ implies $\mathfrak{J} \models \alpha$. Note that *rigid concept names* can be defined in a similar fashion as rigid role names, and can be simulated with $A \sqsubseteq \bigcirc A$ and $A \sqsubseteq \bigcirc^{-1} A$ in $\mathcal{TEL}^{\bigcirc}$ and $\mathcal{TEL}^{\bigcirc}_{lin}$ (but not in $\mathcal{TEL}^{\bigcirc}_{fut}$).

Query answering and ultimate periodicity. The *temporal atomic query answering* (TAQA) problem is that of deciding, given a temporal KB $(\mathcal{T}, \mathcal{A})$ and a fact $A(a, n)$, whether $(\mathcal{T}, \mathcal{A}) \models A(a, n)$. We consider *combined complexity*, where the size of the input is $|\mathcal{T}| + |\mathcal{A}| + |A(a, n)|$, and *data complexity*, where $\mathcal{T}$ is fixed. For the latter, TAQA is PSPACE-complete when the TBox $\mathcal{T}$ is *ultimately periodic*, i.e. when the sets $\{n \in \mathbb{Z} \mid \mathcal{T} \models A \sqsubseteq \bigcirc^n B\}$ are semilinear, for all $A, B \in \mathsf{N_C}(\mathcal{T})$ [Gutiérrez-Basulto *et al.*, 2016].

Conjunctive grammars. To analyse ultimate periodicity of $\mathcal{TEL}^{\bigcirc}$-TBoxes, we employ conjunctive grammars. As introduced by Okhotin [2013], a *conjunctive grammar* is a triple $G = (N, \Sigma, R)$, where N and Σ are disjoint alphabets of *nonterminals* and *terminals*, respectively, and R is a finite set of *rules* of the form:

$$\mathcal{N} \quad \to \quad \alpha_1 \,\&\, \cdots \,\&\, \alpha_n$$

with $\mathcal{N} \in N$, $n \geqslant 1$, and $\alpha_i \in (N \cup \Sigma)^*$. Several rules with the same left-hand side $\mathcal{N}$ can be written as a single rule, with $|$ separating the right-hand sides.

The semantics extends that of context-free grammars with *intersection*: given a rule of the above form, apply "in parallel" context-free rules $\mathcal{N} \to \alpha_i$ and take the intersection of the generated languages. We write $L_G(\mathcal{N})$ for the language generated by the nonterminal $\mathcal{N} \in N$. Languages that can be generated in this way, e.g. $\{a^n b^n c^n \mid n \in \mathbb{N}\}$, are called conjunctive. We refer to Okhotin [2013] for details.

The membership problem for conjunctive languages is PTIME-complete, both when the grammar is fixed or is part of input itself [Okhotin, 2003; 2011]. We will focus on unary grammars, where the underlying alphabet contains just one symbol, i.e. $\Sigma = \{c\}$. A unary language L can be seen as a set of natural numbers given in unary: $\{n \in \mathbb{N} \mid c^n \in L\}$. While unary context-free grammars can generate only regular languages (i.e., semilinear subsets of $\mathbb{N}$) [Parikh, 1966], conjunctive grammars are more powerful. For example, there is a unary conjunctive language corresponding to the set $\{4^n \mid n \in \mathbb{N}\}$ [Jeż, 2008]. Moreover, emptiness and regularity of the language are both undecidable for unary conjunctive grammars [Jeż and Okhotin, 2010].

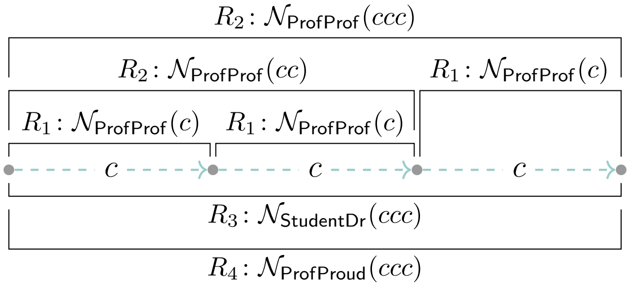

Figure 2: An illustration for the facts that $c^3 \in L_{G_\mathcal{T}}(\mathcal{N}_{\mathsf{ProfProf}})$, in the upper part, and $c^3 \in L_{G_\mathcal{T}}(\mathcal{N}_{\mathsf{ProfProud}})$, in the lower part. Each symbol c stands for a step forward in time. The grammar rule of $G_\mathcal{T}$ used to obtain each proposition is given in the following denotation: $R_1 : \mathcal{N}_{\mathsf{ProfProf}} \to c$, $R_2 : \mathcal{N}_{\mathsf{ProfProf}} \to \mathcal{N}_{\mathsf{ProfProf}}\mathcal{N}_{\mathsf{ProfProf}}$, $R_3 : \mathcal{N}_{\mathsf{StudentDr}} \to c^3$, $R_4 : \mathcal{N}_{\mathsf{ProfProud}} \to \mathcal{N}_{\mathsf{StudentDr}}$.

3 From TBoxes to Grammars, and Back Again

We present two theorems that establish our key result: the sets of the form $\{n \in \mathbb{N} \mid \mathcal{T} \models A \sqsubseteq \bigcirc^n B\}$ for $\mathcal{TEL}^{\bigcirc}_{fut}$ -TBoxes correspond exactly to unary conjunctive languages. In this extended abstract, we assume that all role names in TBoxes are rigid (all the results hold in the general case).

Theorem 2. *For every $\mathcal{TEL}^{\bigcirc}_{fut}$-TBox $\mathcal{T}$, one can construct in polynomial time a unary conjunctive grammar $G_\mathcal{T} = (N, \{c\}, R)$ such that for any $A, B \in \mathsf{N_C}(\mathcal{T})$, there is $\mathcal{N}_{AB} \in N$ such that $c^n \in L_{G_\mathcal{T}}(\mathcal{N}_{AB})$ iff $\mathcal{T} \models A \sqsubseteq \bigcirc^n B$.*

We sketch the construction. Given a $\mathcal{TEL}^{\bigcirc}_{fut}$-TBox $\mathcal{T}$, let $G_\mathcal{T} = (N_\mathcal{T}, \{c\}, R_\mathcal{T})$, where $N_\mathcal{T} = \{\mathcal{N}_{AB} \mid A, B \in \mathsf{N_C}(\mathcal{T})\}$ and $R_\mathcal{T}$ contains exactly:

$$\mathcal{N}_{AB} \to \varepsilon, \qquad \text{for } A \sqsubseteq B \in \mathcal{T} \text{ or } A = B \tag{2}$$

$$\mathcal{N}_{AB} \to c^n, \qquad \text{for } A \sqsubseteq \bigcirc^n B \in \mathcal{T},\ n > 0 \tag{3}$$

$$\mathcal{N}_{AB} \to \mathcal{N}_{AC} \mathbin{\&} \mathcal{N}_{AD}, \qquad \text{for } \begin{array}{l} A \in \mathsf{N_C}(\mathcal{T}), \\ C \sqcap D \sqsubseteq B \in \mathcal{T} \end{array} \tag{4}$$

$$\mathcal{N}_{AB} \to \mathcal{N}_{CD}, \qquad \text{for } \left\{ \begin{array}{c} A \sqsubseteq \exists r.C \\ \exists r.D \sqsubseteq B \end{array} \right\} \subseteq \mathcal{T} \tag{5}$$

$$\mathcal{N}_{AB} \to \mathcal{N}_{AC}\,\mathcal{N}_{CB}, \qquad \text{for } A, B, C \in \mathsf{N_C}(\mathcal{T}) \tag{6}$$

Intuitively, for every pair of concept names $A, B \in \mathsf{N_C}(\mathcal{T})$, $G_\mathcal{T}$ encodes every possible way of deriving $B(a, n)$ from $A(a, 0)$ with respect to $\mathcal{T}$: either directly (2, 3), or by obtaining $C(a, n)$ and $D(a, n)$ that together give $B(a, n)$ (4), or by going through a null (5), or through an intermediate point $C(a, k)$, $0 \leqslant k \leqslant n$ (6). To illustrate, recall the TBox $\mathcal{T}$ from Example 1, which is such that $\mathcal{T} \models \mathsf{Prof} \sqsubseteq \bigcirc^3\mathsf{Happy}$. Figure 2 shows the derivations witnessing $c^3 \in L_{G_\mathcal{T}}(\mathcal{N}_{\mathsf{ProfProf}})$ and $c^3 \in L_{G_\mathcal{T}}(\mathcal{N}_{\mathsf{ProfProud}})$. We obtain $c^3 \in L_{G_\mathcal{T}}(\mathcal{N}_{\mathsf{ProfHappy}})$ using the rule $\mathcal{N}_{\mathsf{ProfHappy}} \to \mathcal{N}_{\mathsf{ProfProf}} \mathbin{\&} \mathcal{N}_{\mathsf{ProfProud}}$.

We now turn our attention to the converse translation.

Theorem 3. *For every unary conjunctive grammar $G = (N, \{c\}, R)$, one can construct in polynomial time a $\mathcal{TEL}^{\bigcirc}_{fut}$-TBox $\mathcal{T}_G$ and $A \in \mathsf{N_C}(\mathcal{T}_G)$, such that for every $\mathcal{B} \in N$ there is $B \in \mathsf{N_C}(\mathcal{T}_G)$ such that $\mathcal{T}_G \models A \sqsubseteq \bigcirc^n B$ iff $c^n \in L_G(\mathcal{B})$.*

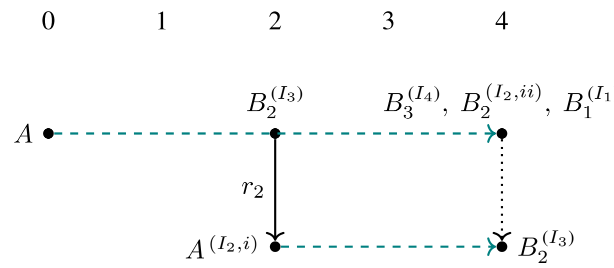

Figure 3: An illustration for $\mathcal{T}_G \models A \sqsubseteq \bigcirc^4 B_1$. The concept inclusion used for each inference is given in superscript (cf. Example 4).

Let $G = (\{\mathcal{B}_1, \dots, \mathcal{B}_m\}, \{c\}, R)$ be a unary conjunctive grammar. W.l.o.g., we assume that its rules are of the forms

$$\mathcal{N} \to w \qquad \mathcal{N} \to \mathcal{N}_1\mathcal{N}_2 \qquad \mathcal{N} \to \mathcal{N}_1 \& \mathcal{N}_2$$

with $w \in \{c\}^*$ and $\mathcal{N}, \mathcal{N}_1, \mathcal{N}_2 \in \{\mathcal{B}_1, \dots, \mathcal{B}_m\}$. Indeed, every unary grammar can be converted to this form in polynomial time. Fix concept names $A, B_1, \dots, B_m$, and rigid role names $r_1, \dots, r_m$. We define $\mathcal{T}_G$ to contain exactly the following concept inclusions, according to the rules of R. [1]

$$A \sqsubseteq B_i, \qquad \text{for each rule } \mathcal{B}_i \to \varepsilon \tag{7}$$

$$A \sqsubseteq \bigcirc^n B_i, \qquad \text{for each rule } \mathcal{B}_i \to c^n, n > 0 \tag{8}$$

$$B_j \sqcap B_k \sqsubseteq B_i, \qquad \text{for each rule } \mathcal{B}_i \to \mathcal{B}_j \& \mathcal{B}_k \tag{9}$$

$$B_j \sqsubseteq \exists r_j.A, \qquad \text{for } 1 \leq j \leq m \tag{10}$$

$$\exists r_j.B_k \sqsubseteq B_i, \qquad \text{for each rule } \mathcal{B}_i \to \mathcal{B}_j\mathcal{B}_k \tag{11}$$

It can be shown that $c^n \in L_G(\mathcal{B}_i)$ iff $\mathcal{T}_G \models A \sqsubseteq \bigcirc^n B_i$. We first illustrate this on an example.

Example 4. *Consider the grammar G defined by the following rules and the corresponding concept inclusions of $\mathcal{T}_G$:*

$$\mathcal{B}_1 \to \mathcal{B}_2 \& \mathcal{B}_3 \qquad B_2 \sqcap B_3 \sqsubseteq B_1 \tag{I_1}$$

$$\mathcal{B}_2 \to \mathcal{B}_2\mathcal{B}_2 \qquad B_2 \sqsubseteq \exists r_2.A\ (i),\ \exists r_2.B_2 \sqsubseteq B_2\ (ii) \tag{I_2}$$

$$\mathcal{B}_2 \to cc \qquad A \sqsubseteq \bigcirc^2 B_2 \tag{I_3}$$

$$\mathcal{B}_3 \to cccc \qquad A \sqsubseteq \bigcirc^4 B_3 \tag{I_4}$$

Then, we have $\mathcal{T}_G \models A \sqsubseteq \bigcirc^4 B_1$, as shown in Figure 3.

More generally, consider a rule of the form $\mathcal{B}_i \to \mathcal{B}_j\mathcal{B}_k$. It states that $c^n \in L_G(\mathcal{B}_i)$ if $n = n_1 + n_2$, $c^{n_1} \in L_G(\mathcal{B}_j)$ and $c^{n_2} \in L_G(\mathcal{B}_k)$. This corresponds to the following line of reasoning with respect to $\mathcal{T}_G$. Starting from $A(a, 0)$, we derive $B_j(a, n_1)$, and then go, by $B_j \sqsubseteq \exists r_j.A$ (10), to a new element, b, where the process restarts, recursively, from $A(b, n_1)$. When $B_k(b, n_1 + n_2)$ is inferred, and only at that point, it is lifted up to $B_i(a, n_1 + n_2)$ by $\exists r_j.B_k \sqsubseteq B_i$ (11).

Linear $\mathcal{TEL}^{\bigcirc}$ and Context-Free Grammars

If a TBox $\mathcal{T}$ belongs to both $\mathcal{TEL}^{\bigcirc}_{fut}$ and $\mathcal{TEL}^{\bigcirc}_{lin}$, the grammar $G_\mathcal{T}$ provided by Theorem 2 does not contain rules of type (4), and is thus context-free. This connection can be extended to the linear fragment in general.

[1] We present here a simplified construction compared to Bourgaux *et al.* [2025a], assuming a stronger normal form of G.

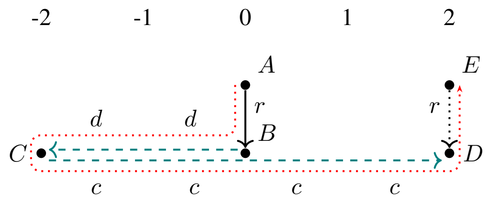

Figure 4: The word $ddcccc$ read along the dotted line witnesses that $\mathcal{T} \models A \sqsubseteq \bigcirc^2 E$.

Theorem 5. *For every $\mathcal{TEL}^{\bigcirc}_{lin}$-TBox $\mathcal{T}$, there exists a context-free grammar $\Gamma_{\mathcal{T}} = (N, \{c, d\}, R)$, of size polynomial in $|\mathcal{T}|$, such that for any $A, B \in \mathsf{N_C}(\mathcal{T})$, there is $\mathcal{N}_{AB} \in N$ such that $\mathcal{T} \models A \sqsubseteq \bigcirc^n B$ iff there exists $w \in L_{\Gamma_{\mathcal{T}}}(\mathcal{N}_{AB})$ with $\#c(w) - \#d(w) = n$.*

In a word $w \in \{c, d\}^*$, a symbol c corresponds to a step forwards in time, and a symbol d to a step backwards. Otherwise, the intuition behind $\Gamma_{\mathcal{T}}$ is the same as that given for $G_{\mathcal{T}}$ from Theorem 2.

Example 6. *Consider a $\mathcal{TEL}^{\bigcirc}_{lin}$-TBox $\mathcal{T}$ containing the following concept inclusions, with $r \in \mathsf{N_R^{rig}}$. Figure 4 shows a word witnessing that $\mathcal{T} \models A \sqsubseteq \bigcirc^2 E$.*

$$A \sqsubseteq \exists r.B \qquad B \sqsubseteq \bigcirc^{-2} C \qquad C \sqsubseteq \bigcirc^4 D \qquad \exists r.D \sqsubseteq E$$

Note that Theorem 5 is not constructive. For a $\mathcal{TEL}^{\bigcirc}_{lin\&rig}$-TBox, however, one can compute $\Gamma_{\mathcal{T}}$ in polynomial time, just like in Theorem 2.

4 Answering Temporal Atomic Queries

We use Theorems 2, 3, and 5 to analyse ultimate periodicity of TBoxes and derive complexity bounds for TAQA.

First, we close the question of $\mathcal{TEL}^{\bigcirc}$ ultimate periodicity.

Theorem 7. *The following statements hold.*

(i) There is a $\mathcal{TEL}^{\bigcirc}_{fut}$-TBox which is not ultimately periodic.

(ii) It is undecidable to check if $\{n \in \mathbb{N} \mid \mathcal{T} \models A \sqsubseteq \bigcirc^n B\}$ is semilinear for a $\mathcal{TEL}^{\bigcirc}_{fut}$-TBox $\mathcal{T}$ and $A, B \in \mathsf{N_C}$.

Proof sketch. Point *(i)* holds by Theorem 3 and the existence of a non-regular unary conjunctive language [Jeż, 2008]. For *(ii)*, we give a reduction from the problem of deciding whether $L_G(\mathcal{B})$ is regular for a unary conjunctive grammar G and a nonterminal $\mathcal{B}$, which is undecidable [Jeż and Okhotin, 2010]. $\square$

Second, by a reduction from the undecidable problem of emptiness checking for unary conjunctive languages [Jeż and Okhotin, 2010], we obtain undecidability results for TAQA.

Theorem 8. *TAQA is undecidable, for combined complexity, with $\mathcal{TEL}^{\bigcirc}$-TBoxes and with $\mathcal{TEL}^{\bigcirc}_{fut}$-TBoxes extended with rigid concept names.*

We obtain positive results as well. First, we show that TAQA is decidable with $\mathcal{TEL}^{\bigcirc}_{fut}$-TBoxes.

Theorem 9. *TAQA with $\mathcal{TEL}^{\bigcirc}_{fut}$-TBoxes is* PTIME*-complete, both for combined and data complexity.*

Proof sketch. The lower bounds hold already for the description logic $\mathcal{EL}$ (without temporal operators) [Calvanese *et al.*, 2013]. For the upper bounds, one can provide a polynomial reduction from the problem of deciding whether $(\mathcal{T}, \mathcal{A}) \models A(a, n)$ to that of checking whether a word belongs to the language of a conjunctive grammar, which can be tested in polynomial time [Okhotin, 2003; 2011]. $\square$

We note that Theorem 9 can be alternatively proved by a reduction to reasoning with *temporally acyclic* TBoxes [Gutiérrez-Basulto *et al.*, 2016]. However, the connection to grammars allows us to reuse efficient algorithms and tools [Okhotin, 2002; Okhotin and Reitwießner, 2012].

We now turn to the linear fragment. For an ultimately periodic $\mathcal{T}$, let $\|\mathcal{T}\|$ denote the largest *norm* among the (semilinear) sets $\{n \in \mathbb{Z} \mid \mathcal{T} \models A \sqsubseteq \bigcirc^n B\}$ [Chistikov and Haase, 2016].

Theorem 10. *The following statements hold.*

(i) Every $\mathcal{TEL}^{\bigcirc}_{lin}$-TBox $\mathcal{T}$ is ultimately periodic, $\|\mathcal{T}\| \leqslant 2^{\mathrm{poly}(|\mathcal{T}|)}$.

(ii) TAQA with $\mathcal{TEL}^{\bigcirc}_{lin}$-TBoxes is NL*-complete, for data complexity.*

(iii) TAQA with $\mathcal{TEL}^{\bigcirc}_{lin\&rig}$-TBoxes is in EXPSPACE*, for combined complexity.*

Proof sketch. The point *(i)* follows from Theorem 5 and results on context-free languages [Parikh, 1966; Esparza *et al.*, 2011]. The lower bound in *(ii)* follows from the atemporal case [Calvanese *et al.*, 2013]. For the upper bounds, both in *(ii)* and *(iii)*, one can translate $\mathcal{TEL}^{\bigcirc}_{lin}$-TBoxes to programs of linear temporal datalog [Artale *et al.*, 2025], following the general approach of [Gutiérrez-Basulto *et al.*, 2016] to ultimately periodic TBoxes. $\square$

Following Gutiérrez-Basulto *et al.* [2016], we assume a unary encoding of numbers. For the case of binary encoding, and for the missing bounds (data complexity for $\mathcal{TEL}^{\bigcirc}$, combined complexity for $\mathcal{TEL}^{\bigcirc}_{lin}$ and a lower bound for $\mathcal{TEL}^{\bigcirc}_{lin\&rig}$), we refer the reader to Gnatenko and Kontchakov [2026].

5 Conclusions

Connections between temporal logics, such as LTL, and formal languages (in particular, regular languages) are well-known [Vardi, 1996; Demri *et al.*, 2016]. We establish new ones, between the linear and future fragments of the temporal description logic $\mathcal{TEL}^{\bigcirc}$, on the one side, and context-free and unary conjunctive languages, on the other. Using these connections, we obtain several important results on $\mathcal{TEL}^{\bigcirc}$, both negative and positive, relying on the formal language theory.

Ethical Statement

There are no ethical issues.

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